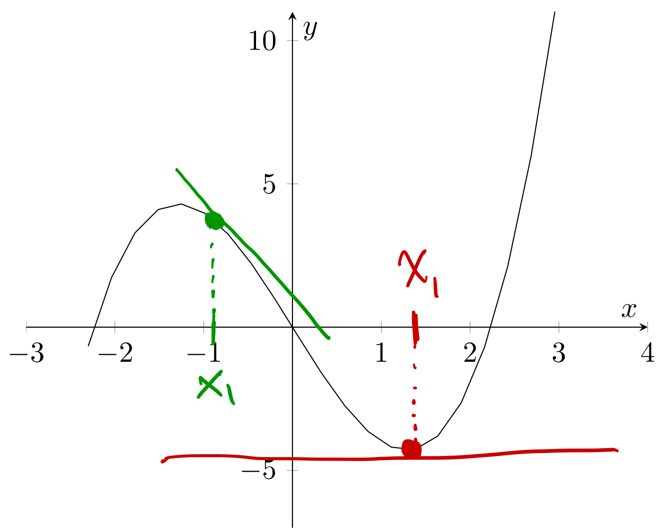
LECTURE NOTES: 4-8 NEWTON'S METHOD (PART 2)

RECALL FROM THE PREVIOUS CLASS:

We used Newton's Method to estimate the positive root of the function $f(x) = x^3 - 5x$. (Graphed below.)



QUESTION 1: What is the "formula" we used in Newton's Method? That is, if x_n is an estimate of a root, how does one calculate the next (better) estimate, x_{n+1}?

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Pick x_1 as an initial guess & Newton's method will converge to the "wrong" root

QUESTION 2: Yesterday, we began our estimation by "guessing" the root was about 3. (That is, we chose $x_1 = 3$.) How much did this choice matter? Are there any truly *bad* guesses or will any guess eventually get us to the desired root? Explain your conclusion.

There are bad choices. Examples: Point P has a horizontal tangent. So picking x_1 would result in <u>NO</u> x-intercept (or zero for $f'(x_n)$).

QUESTION 3: What sort of conditions do you think need to hold in order to make Newton's Method work and work properly?

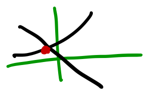
- a good starter guess.
- a function that is differentiable & continuous near the root in question.

PRACTICE PROBLEMS: For each problem below, use Newton's Method to answer the question. Explain how you chose x_1. Use your calculator to graph the function + identify the root(s) you approximated.

1. Approximate any zeros of $f(x) = e^x + x$. using 3 iterations of Newton's Method.

thinking: $0 = e^x + x$ is equivalent to $e^x = -x$

Geometrically,



I can conclude there is one, negative root.

Since $f(0) = 1 > 0$ and $f(-1) = -1 + \frac{1}{e} < 0$,

I will pick $x_1 = -\frac{1}{2}$.

Since $f'(x) = e^x + 1$, Newton's Method will use:

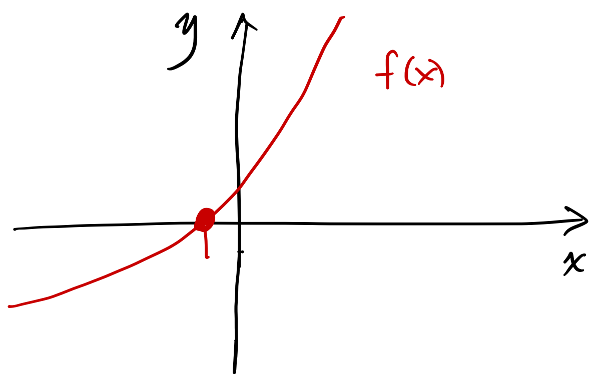
$$x_{n+1} = x_n - \frac{e^{x_n} + x_n}{e^{x_n} + 1}$$

If $x_1 = -\frac{1}{2}$, then $x_2 = -\frac{1}{2} - \dfrac{e^{-1/2} - \frac{1}{2}}{e^{-1/2} + 1} \approx -0.5663110$

So $x_3 = x_2 - \dfrac{e^{x_2} + x_2}{e^{x_2} + 1} \approx -0.567143$

whoa!

$x_4 = -0.567143$



2. Approximate any zeros of $g(x) = x - 2\sin x$. accurate to at least 9 decimal places.

Clearly $x=0$ is a zero of $g(x)$. Also, since $g(x)$ is an odd function, we know that if we find a positive root we will find the symmetric negative root simultaneously.

$g\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - 2 < 0$

$g(\pi) = \pi - 2\sin(\pi) = \pi > 0$

So there is a root between 1.5 and 3.

Pick $x_1 = 2$. Newton's Formula here is:

$$x_{n+1} = x_n - \frac{x_n - 2\sin x_n}{1 - 2\cos x_n}$$

$x_1 = 2$

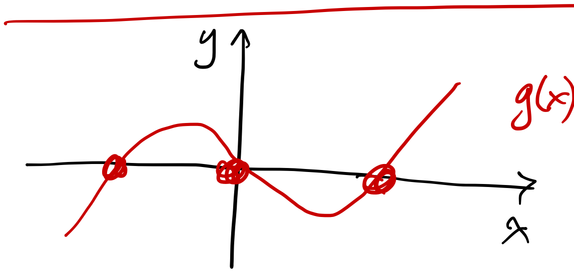
$x_2 = 2 - \dfrac{2 - 2\sin(2)}{1 - 2\cos(2)} \approx 1.90099594$

$x_3 \approx 1.895511645$

$x_4 \approx 1.895494267$

$x_5 = x_4$.

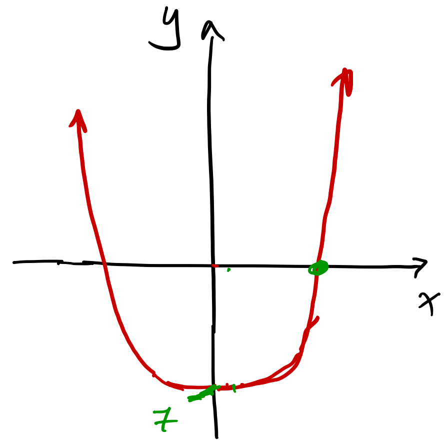
$\boxed{\text{So } x = \pm 1.895494267, 0 \text{ are zeros of } g(x).}$



3. Estimate $\sqrt[6]{7}$ correct to 5 decimal places. [Note: You need to construct the "$f(x)$" here.]

$f(x) = x^6 - 7$ will have $\sqrt[6]{7}$ as a root

$f(1) = -6 < 0$, $f(2) > 0$. I'll pick $x_1 = 1.1$



Newton's Formula would be:

$$x_{n+1} = x_n - \frac{(x_n)^6 - 7}{6x_n^5} = x_n - \frac{1}{6}x_n + \frac{7}{6}x_n^{-5}$$

$$= \frac{1}{6}\left(5x_n + 7x_n^{-5}\right)$$

If $x_1 = 1.1$, then $x_2 = \frac{1}{6}\left(5(1.1) + 7(1.1)^{-5}\right)$

≈ 1.641074877

$x_3 \approx 1.465580165$

$x_4 \approx 1.39386042$

$x_5 \approx 1.383293572$

$x_6 \approx \underline{1.383087}631$

$x_7 \approx \underline{1.383087}554$

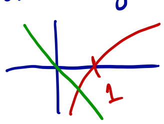
4. Approximate the x-value of the point of intersection of $f(x) = \frac{-x}{3}$ and $g(x) = \ln x$. Continue the process until two successive approximations differ by less than 0.001

We want x so that $\frac{-x}{3} = \ln x$.

Alternately, we want roots to

$H(x) = \frac{1}{3}x + \ln x$.

From the graphs of $f(x)$ and $g(x)$



we know the point of intersection must be between $x=0$ and $x=1$.

Choose $x_1 = \frac{1}{2}$.

$$x_{n+1} = x_n - \frac{\frac{1}{3}x_n + \ln x_n}{\frac{1}{3} + \frac{1}{x_n}} \cdot \frac{3x_n}{3x_n}$$

$$= x_n - \frac{x_n^2 + 3x_n \ln x_n}{x_n + 3}$$

If $x_1 = \frac{1}{2}$, $x_2 = 0.725634506$.

$x_3 \approx 0.7716955824$

$x_4 \approx 0.7728822333$

$x_5 \approx 0.7728829591 = x_6$